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NUMERICAL SOLUTION OF PLANE AND AXIALLY SYMMETRIC JET FLOW PROBLEMS BASED ON THE VARIATIONAL INEQUALITY FORMULATION

HAN-YONG LEE AND MOON-UHN KIM

Department of Applied Mathematics, Korea Advanced Institute of Science and Technology, PO Box 150, Cheongryang, Seoul, Korea

SUMMARY

The numerical analysis of plane and axially symmetric jet flows of an incompressible inviscid fluid is treated. A new formulation of the variational inequality type is developed from the variational principle associated with jet problems. A successive approximation method is formulated by the combined use of variational inequality and the finite element method. Numerical examples based on the iterative method are presented. The results obtained agree well with those by other methods.

KEY WORDS Jet flow Variational inequality Finite element method

1. INTRODUCTION

This paper considers the jet flow of an incompressible inviscid fluid which is moving in a nozzle and is exiting from a prescribed small opening. As is well known, the jet problem is difficult to treat either analytically or numerically because the location of the free boundary and the speed of the fluid on the free boundary are not known *a priori.*

Theoretical aspects (existence, uniqueness, etc.) for two-dimensional and axially symmetric jet flows have been extensively studied in past years. Alt *et al.*[1,2] and Friedman[3] have recently established existence and uniqess theorems for axially symmetric and two-dimensional asymmetric jet flows under rather weaker assumptions on the nozzle by formulating as variational problems with parameters. Their results naturally apply to plane symmetric jet flows. We summarize some of their results relevant to our analysis in Section 2.

Various numerical methods for jet flows have also been developed; excellent surveys are given in the monographs of Gilbarg[4] and Gurevich.[5] Recently, Aitchison has reported a method for the numerical solutions of a plane symmetric jet flow[6] and plane and axially symmetric finite cavity flows.[7] He has formulated the problem as the minimization of a functional over a variable domain, which has been solved by the method of variable finite elements.

In this paper a new numerical algorithm for plane and axially symmetric jet flows is proposed. On the basis of the results of References 1–3, an alternative formulation of the variational inequality type is developed by taking the first variation of the functional at the stationary point. The variational inequality formulation naturally yields successive approximations which can be implemented by the conventional finite element method, though the domain of integration is varying at each iteration.

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2. PLANE AND AXIALLY SYMMETRIC JET FLOWS

2.1. Statement of problems

We consider the plane (or axially) symmetric flow of an incompressible inviscid fluid issuing from the prescribed small opening of a nozzle. The flow configuration is shown in Figure 1, where N is the nozzle, Γ is the free boundary and A $= (0, 1)$. We denote by Ω the flow domain bounded by N, Γ and the x-axis.

For a given flux Q, the plane symmetric jet problem is to find a streamfunction u and a free boundary Γ such that u satisfies

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega \tag{1}$$

subject to

$$u(x, 0) = 0 \qquad \text{for } -\infty < x < \infty, \tag{2}$$

$$u = Q \qquad \text{on N and } \Gamma, \tag{3}$$

$$\frac{\partial u}{\partial n} = \lambda \qquad \text{on } \Gamma, \tag{4}$$

where n is the outward normal to Γ and λ is an unknown constant to be determined as a part of the solution.

It is assumed that the nozzle satisfies all the conditions described in Alt *et al.*[1,2] Then the parameter λ can be assumed as $\lambda \geqslant Q$.

The axially symmetric jet problem is described by replacing (1) and (4) with

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{y}\frac{\partial u}{\partial y} = 0 \quad \text{in } \Omega \tag{5}$$

and

$$\frac{1}{y}\frac{\partial u}{\partial n} = \lambda \quad \text{on } \Gamma \tag{6}$$

respectively. For the axially symmetric flow it is assumed that $\lambda \geqslant 2Q$.

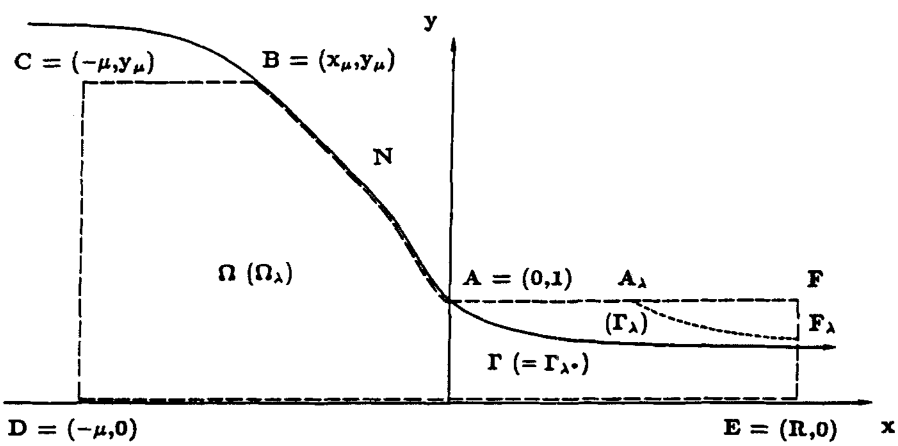

Figure 1. The flow configuration (The truncated domain $\Omega_{\mu, R}$ is presented by the region bounded by dashed lines.)

2.2. Associated variational problems with a parameter

For the variational formulation associated with the jet problem we need a truncated domain $\Omega_{\mu,R}$ with sufficiently large μ and R bounded by ABCDEF as shown in Figure 1. In the plane symmetric case we consider the variational problem of minimizing a functional

$$J_\lambda(v) = \int_{\Omega_{\mu,R}} |\nabla v - \lambda I_{\{v<Q\}} \mathbf{e}|^2 \, dx\, dy \qquad (\mathbf{e} = (0, 1)) \tag{7}$$

over a class of admissible functions $K_{\mu,R}$, where $K_{\mu,R}$ consists of functions v in $H^1(\Omega_{\mu,R})$ satisfying

$$0 \leqslant v \leqslant Q \quad \text{in } \Omega_{\mu,R}, \tag{8}$$

$$v = u_\mu \quad \text{on } \widehat{\mathrm{AE}}, \tag{9}$$

$$v = Q \quad \text{on AF}, \tag{10}$$

$$v = \min\{\lambda y, Q\} \quad \text{on EF}. \tag{11}$$

Here $I_{\{v<Q\}}$ is the indicator function defined by

$$I_{\{v<Q\}}(x, y) = \begin{cases} 1, & \text{if } v(x, y) < Q, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

$H^1(\Omega_{\mu,R})$ is the Sobolev space (of order one) of functions v on $\Omega_{\mu,R}$, and u_μ on $\widehat{\mathrm{AE}}$ (the part of the boundaries consisting of AB, BC, CD and DE) is defined by

$$u_\mu = \begin{cases} Q & \text{on AB and BC}, \\ Qy/y_\mu & \text{on CD}, \\ 0 & \text{on DE}. \end{cases} \tag{13}$$

Functional (7) may be written as a functional over a variable domain,

$$J_\lambda(v) = \int_{\Omega_{\mu,R} \cap \{v<Q\}} |\nabla v - \lambda \mathbf{e}|^2 \, dx\, dy \quad \text{for } v \in K_{\mu,R}, \tag{14}$$

since $v \in K_{\mu,R}$ takes the constant value Q when $v(x, y) < Q$ is not satisfied, and it is noted that

$$\nabla v = 0 \quad \text{a.e. in } \{v = Q\}. \tag{15}$$

Alt *et al.*[1,2,3] have shown that the functional J_λ has a minimizer for any λ and that for some $\lambda = \lambda(\mu, R)$ the minimizer solves the jet problem in the truncated domain $\Omega_{\mu,R}$. We summarize some of their results relevant to our subsequent analysis.

1. The minimization problem (7) has a unique solution $u = u_\lambda$ in $K_{\mu,R}$ such that

$$\nabla^2 u = 0 \quad \text{in } \Omega_\lambda, \tag{16}$$

$$\frac{\partial u}{\partial n} = \lambda \quad \text{on } \Gamma_\lambda, \tag{17}$$

where Ω_λ is the bounded domain in $\Omega_{\mu,R}$ satisfying $u < Q$. The free boundary Γ_λ is analytic and is given by $x = k_\lambda(y)$. The starting point of the free boundary Γ_λ is denoted by $\mathrm{A}_\lambda = (x_\lambda, 1)$, where $x_\lambda = k_\lambda(1)$ (see Figure 1).

2. If $\lambda > Q$ and $\lambda - Q$ is small enough then $x_\lambda > 0$ and

$$\frac{\partial u}{\partial y} \geqslant \lambda \quad \text{on } y = 1, 0 < x < x_\lambda. \tag{18}$$

3. If $\lambda_1 > \lambda_2$ then $x_{\lambda_1} < x_{\lambda_2}$ and Γ_{λ^1} lies below Γ_{λ^2}.
4. If we denote by λ^* the supremum of parameters λ satisfying $x_\lambda \geqslant 0$, then $(u_{\lambda^*}, \Gamma_{\lambda^*}, \lambda^*)$ is the unique solution of the jet problem in the truncated domain $\Omega_{\mu, R}$ such that the free boundary Γ_{λ^*} initiates at A = (0, 1) and its initial direction coincides with that of the nozzle at A.

Thus the plane symmetric jet problem in $\Omega_{\mu, R}$ is equivalent to the problem of finding the minimizer u_{λ^*} of the functional J_λ and a positive parameter λ^*. Alt *et al.*[1,2] have shown that the desired solution of the jet problem in the domain Ω can be obtained by letting $R \to \infty$ and then $\mu \to \infty$.

For the numerical implementation, however, minimization of the functional J_λ given by (7) is rather inconvenient since J_λ contains the term $I_{\{v < Q\}}$. In the following we give an alternative formulation which is obtained by taking the first variation of J_λ given by (14) at the stationary point.

For brevity of description let $u = u_\lambda$ be a minimizer of J_λ and $\Gamma = \Gamma_\lambda$ be the associated free boundary. In order to obtain the first variation of J_λ over a variable domain, following Courant and Hilbert[8] we introduce the family of transformations on $\Omega_{\mu, R}$

$$x^* = X(x, y; \varepsilon), \qquad y^* = Y(x, y; \varepsilon), \tag{19}$$

depending on a parameter $\varepsilon (0 \leqslant \varepsilon \leqslant 1)$. We assume that this transformation is one-to-one continuously differentiable and reduces to the identity transformation for $\varepsilon = 0$. For any point $(x, y) \in \Gamma$, in particular, x^* and y^* are defined by

$$x^* = x + \varepsilon \alpha n_x, \qquad y^* = y + \varepsilon \alpha n_y \tag{20}$$

for a test function α given on Γ, where $\mathbf{n} = (n_x, n_y)$ is the outward normal to Γ. Since Γ is analytic, both αn_x and αn_y are also test functions. The terms $\varepsilon \alpha n_x$ and $\varepsilon \alpha n_y$ in (20) can be interpreted as the variations δx and δy on Γ in Reference 8 respectively.

We assign to the point (x, y) in the old co-ordinates a new function value

$$u^*(x, y; \varepsilon) = u(x, y) + \varepsilon \zeta(x, y) \tag{21}$$

for a test function ζ given on $\Omega_{\mu, R}$. For any point $(x, y) \in \Gamma$ we define u^* by

$$u^*(x^*, y^*; \varepsilon) = u(x, y). \tag{22}$$

Substitution of (20) into (22), to the first order in ε, yields

$$\zeta + \alpha \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{23}$$

In obtaining (23), relation (21) has been used.

In line with Courant and Hilbert it can be shown that the first variation δJ_λ of (14) is given by

$$\delta J_\lambda = \varepsilon \int_{\Omega_{\mu, R} \cap \{u < Q\}} (-2\nabla^2 u)\zeta \, \mathrm{d}x \, \mathrm{d}y + \varepsilon \int_\Gamma 2(\nabla u - \lambda \mathbf{e}) \cdot \mathbf{n} \zeta \, \mathrm{d}s + \varepsilon \int_\Gamma |\nabla u - \lambda \mathbf{e}|^2 \alpha \, \mathrm{d}s \tag{24}$$

for all test functions ζ, where the first term on the right-hand side is taken in the distributional sense.

By Green's theorem, (24) becomes

$$\delta J_{\lambda}=\varepsilon\int_{\Omega_{\mu,R}\cap\{u<Q\}}2\nabla\mathbf{u}\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}y+\varepsilon\int_{\Gamma}2\lambda^{2}\alpha\,\mathrm{d}s+\varepsilon\int_{\Gamma}2\lambda\left(\frac{\partial u}{\partial y}\alpha-n_{y}\zeta\right)\mathrm{d}s. \quad (25)$$

Noting that the condition $u|_{\Gamma}=Q$ implies $(\partial u/\partial y)\,\mathrm{d}s=-(\partial u/\partial n)\mathrm{d}x$ on Γ, substituting (17) and (23) into (25) yeilds

$$\delta J_{\lambda}=\varepsilon\int_{\Omega_{\mu,R}\cap\{u<Q\}}2\nabla u\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}y-\varepsilon\int_{\Gamma}2\lambda\zeta\,\mathrm{d}s. \quad (26)$$

Since $K_{\mu,R}$ is closed and convex, the fact that u is a minimizer of J_{λ} implies

$$\delta J_{\lambda}\geqslant 0. \quad (27)$$

Putting $\zeta=v-u$ for any $v\in K_{\mu,R}$ and taking into consideration (15), inequality (27) can be rewritten as

$$\int_{\Omega_{\mu,R}}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}y\geqslant\int_{\Gamma}\lambda(v-u)\,\mathrm{d}s\quad\text{for any } v\in K_{\mu,R}. \quad (28)$$

It is also to be noted that the solution of inequality problem (28) satisfies (16), (17) and (18).

In the above formulation the integration on the left-hand side can be formally regarded as the variation of J_{λ} for a fixed domain $\Omega_{\mu,R}$, while that on the right-hand side can be ascribed to the variation of the domain. The fact that the variation of J_{λ} is separated into two parts enables us to carry out numerical calculations by using the conventional finite element method.

The inequality problem thus obtained actually falls within the scope of variational inequality problems. However, the variational inequality type of formulation (28) differs from those discussed in References 9–13 in that it contains the term represented by the line integral on the unknown free boundary.

For the axially symmetric jet problem we work with the functional

$$J_{\lambda}(v)=\int_{\Omega_{\mu,R}}\left|\frac{1}{y}\nabla v-\lambda I_{\{v<Q\}}\mathbf{e}\right|^{2}y\,\mathrm{d}x\,\mathrm{d}y \quad (29)$$

and a class of admissible functions $v\in K_{\mu,R}$ given by replacing (11) with

$$v=\min\{\tfrac{1}{2}\lambda y^{2},Q\}\quad\text{on EF}. \quad (30)$$

In the exactly same way, the minimization problem given by (29) is reducible to the variational inequality problem such that the minimizer u satisfies

$$\int_{\Omega_{\mu,R}}\frac{1}{y}\nabla u\cdot\nabla(v-u)\,\mathrm{d}x\,\mathrm{d}y\geqslant\int_{\Gamma}\lambda(v-u)\,\mathrm{d}s\quad\text{for any } v\in K_{\mu,R}. \quad (31)$$

In obtaining (31), the family of transformations (20) is replaced by

$$x^{*}=x+\varepsilon\alpha\frac{n_{x}}{y},\qquad y^{*}=y+\varepsilon\alpha\frac{n_{y}}{y} \quad (32)$$

for a test function α given on Γ.

The solution $u = u_\lambda$ of the inequality problem (31) satisfies

$$Lu = 0 \quad \text{in } \Omega_\lambda, \tag{33}$$

$$\frac{1}{y}\frac{\partial u}{\partial n} = \lambda \quad \text{on } \Gamma_\lambda, \tag{34}$$

$$\frac{1}{y}\frac{\partial u}{\partial y} \geqslant \lambda \quad \text{on} \quad y = 1, 0 < x < x_\lambda. \tag{35}$$

3. NUMERICAL ALGORITHM

In this section a numerical algorithm for solving the jet problem in the truncated domain is presented which is based on the variational inequality formulation obtained in Section 2. The algorithm consists of two parts: one is for finding the location of the free boundary Γ_λ for a given λ and the other is for determining the value of λ^* which ensures that the free boundary detaches from the nozzle.

Since the formulation contains the line integral on the unknown free boundary, the computational domain varies at each iteration. However, the algorithm can be implemented by using the conventional finite element method, in contrast with Aitchison[6, 7] who used the method of variable finite elements.

For brevity let us introduce two functions given by

$$g(x, y) = \begin{cases} 1 & \text{for plane symmetric flow,} \\ 1/y & \text{for axially symmetric flow,} \end{cases} \tag{36}$$

$$u_R(R, y) = \begin{cases} \lambda y & \text{for plane symmetric flow,} \\ \frac{1}{2}\lambda y^2 & \text{for axially symmetric flow.} \end{cases} \tag{37}$$

Following traditional methods we construct an algorithm for finding the free boundary for a given λ: starting from an initial guess Ω^0 for Ω_λ and Γ^0 for Γ_λ we solve the problem (16) and (17) (or (33) and (34)); then Γ^0 is modified into Γ^1 and Ω^0 into Ω^1 such that the condition $u^1 = Q$ is satisfied; finally the procedure is iterated until a satisfactory convergence is obtained. A proposed algorithm is as follows.

Algorithm 1

1. Take an initial domain Ω^0 sufficiently large containing Ω_λ and let $\Gamma^0 = \mathrm{A}_\lambda^0\mathrm{F}_\lambda^0$ be the assumed initial free boundary.
2. For $n \geqslant 1$ find u^n satisfying $u^n = u_\mu$ on $\widehat{\mathrm{AE}}$, $u^n = u_R$ on $\mathrm{EF}_\lambda^{n-1}$ and such that

$$\int_{\Omega^{n-1}} g\nabla u^n \cdot \nabla w \, \mathrm{d}x\,\mathrm{d}y = \int_{\mathrm{AA}_\lambda^{n-1}} \lambda w \, \mathrm{d}x + \int_{\Gamma^{n-1}} \lambda w \, \mathrm{d}s \tag{38}$$

   for all test functions w with $w = 0$ on $\partial\Omega_{\mu, R}\backslash\mathrm{AF}$.
3. Obtain a continuous curve $\Gamma^n = \mathrm{A}_\lambda^n\mathrm{F}_\lambda^n$ in Ω^{n-1} by solving the equation

$$u^n(x, y) = Q. \tag{39}$$

   Then let $n = n + 1$ and go to Step 2. If there exists no curv e in Ω^{n-1} satisfying (39), then we replace the domain Ω^{n-1} with a larger domain $\tilde{\Omega}^{n-1}$ containing Ω^{n-1} and contained in Ω^{n-2} and return to Step 2.

The iteration procedure is continued until Γ^n coincides with Γ^{n-1} within a prescribed error. It is to be noted that F_λ^n is always equal to F_λ for $n \geqslant 1$.

Although Algorithm 1 consists of an iterative procedure, from the performed numerical experiments it has been observed that the first iteration yields a remarkably accurate approximation to the location of the free boundary Γ_λ for each given λ. It is also to be noted that in solving the jet problem there is little need to obtain the exact location of the free boundary for λ other than λ^*.

An algorithm for determining the value of λ^*, which solves the jet problem in the truncated domain $\Omega_{\mu, R}$, is constructed as follows.

Algorithm 2

1. Assume the initial domain $\Omega_{\mu, R}$ and choose λ^1 such that $\lambda^1 - Q$ (or $\lambda^1 - 2Q$) is small enough for plane (or axially) symmetric flow. Algorithm 1 gives u_{λ^1} and $\Gamma_{\lambda^1} = A_{\lambda^1}F_{\lambda^1}$. Since $\lambda^1 - Q$ (or $\lambda^1 - 2Q$) is assumed small enough, the x-co-ordinate x_{λ^1} of A_{λ^1} (the starting point of Γ_{λ^1}) is positive.
2. For $n \geqslant 2$, with a given $\rho > 0$, define λ^n by
$$\lambda^n = \lambda^{n-1} + \rho(\tilde{\lambda}^{n-1} - \lambda^{n-1}), \tag{40}$$
where $\tilde{\lambda}^{n-1} = (\partial u_{\lambda^{n-1}}/\partial y)(\tilde{x}, 1)$ with $0 < \tilde{x} < x_{\lambda^{n-1}}$. Since $x_\lambda \to 0$ as $\lambda \uparrow \lambda^*$ we can take $\tilde{x} = 0$, and then inequality (8) or (35) indicates that $\{\lambda^n\}$ is an increasing sequence.
3. Using Algorithm 1, obtain u_{λ^n} and $\Gamma_{\lambda^n} = A_{\lambda^n}F_{\lambda^n}$. If x_{λ^n} (the x-co-ordinate of A_{λ^n}) $\geqslant 0$ then let $n = n + 1$ and go to Step 2. Otherwise replace the value of ρ with a smaller value (say $\frac{1}{2}\rho$) and return to Step 2.

The iteration is continued until
$$|x_{\lambda^n}| < \varepsilon \tag{41}$$
is satisfied for a prescribed tolerance ε. Properties 1–4 summarized in Section 2 assure the convergence of Algorithm 2.

Taking into account the remarks following Algorithm 1, for practical calculations of the jet problem we can propose a modified algorithm in which the application of Algorithm 1 in Algorithm 2 is replaced with solving (38) and (39) only once. Experimentally we have observed that the modified algorithm provides satisfactory results for a given ρ.

The numerical treatment of the minimization problem of a functional over a variable domain by the method of variable finite elements requires the geometric variables to consider the effect of the variation of the domain.[6, 7] The resulting discretized functional becomes a quadratic function of the nodal values for u but a complicated function of the geometric variables. The method of determining the minimizer of the discrete functional with respect to both nodal values and geometric variables is very involved. On the other hand the numerical solution of equation (38), in which the effect of the variation of the domain is taken into consideration, can be easily obtained by using the conventional finite element method. It is also to be noted that the iterative procedure for determining the value of λ^* is much simpler in Algorithm 2 than that in Aitchison,[6] since the former requires calculation of the velocity at only one point at each iteration while the latter needs to calculate the pressure in the whole flowfield.

4. NUMERICAL EXAMPLES

We apply Algorithm 2 described in the previous section to the plane (or axially) symmetric flows from the opening between semi-infinite straight (or conical) walls (Figure 2) and from a vessel with

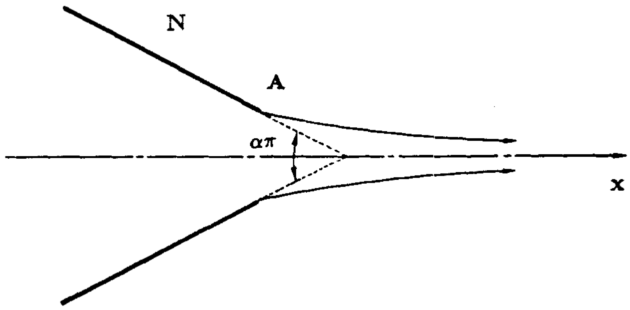

Figure 2. Flow from the opening between semi-infinite straight walls

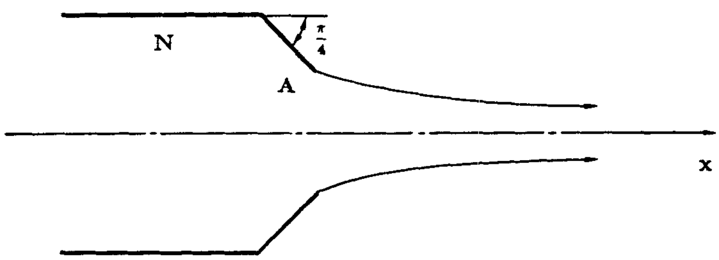

Figure 3. Jet from a vessel with a funnel-shaped bottom

a funnel-shaped bottom (Figure 3). In each case we focus our attention on determining the jet contraction coefficient C, which is defined as the ratio of the cross-sectional area of the jet to that of the opening.

For the numerical computations the truncated initial domain $\Omega_{\mu,R}$ is divided into a large number of quadrilateral elements by using the automatic mesh generation program described in Durocher and Gasper.[14] The initial domain $\Omega_{\mu,R}$ was determined so as to have no significant effect on the flow after several experiments. The streamfunction u is approximated as usual by piecewise linear functions for the plane symmetric case, and for the axially symmetric case by quadratic functions since $u \to \frac{1}{2}\lambda y^2$ as $x \to \infty$.

At each iteration ($n \geqslant 1$) the location of the current free boundary Γ^n is determined by finding the root (x_γ^n, y_γ^n) of the equation

$$u^n(x, y) = Q \tag{42}$$

along the side of the master element using the isoparametric relation. The mesh for the iterated domain Ω^n is generated in such a way that only the y-co-ordinates of nodes placed below Γ^n are scaled proportionally to the height y_λ^n and all the other co-ordinates are kept constant. The linkage structure between elements remains fixed.

Expressions for the discrete approximation of (38) can be written in the form

$$\mathbf{K}^n \mathbf{u}^{n+1} = \mathbf{f}^n \quad \text{for each } n \geqslant 0, \tag{43}$$

where $\mathbf{K}^n$ is a sparse symmetric positive definite matrix, $\mathbf{u}^n = \{u_i^n\}$ are the nodal values and $\mathbf{f}^n$ is the vector resulting from the integral over AA^n and Γ^n. The set of linear equations (43) can be solved by using the band storage scheme and the Cholesky decomposition.[15,16]

4.1. *Flow from the opening between semi-infinite straight walls*

We consider the efflux of a plane jet from an opening in the vessel bounded by semi-infinite straight walls, as shown in Figure 3, assuming the flow to be symmetric with respect to the x-axis. The angle between the walls is $\alpha\pi$ $(0 < \alpha \leqslant 2)$. For $\alpha = 1$ the flow describes a jet issuing from a slot in the plane wall and the case $\alpha = 2$ corresponds to Borda's mouthpiece. The analytic results for the contraction coefficient C are available:[4]

$$\frac{1}{C} = 2 - \frac{1}{\pi}\sin\frac{\alpha\pi}{2}\left[\Psi\left(\frac{1}{2}+\frac{\alpha}{4}\right) - \Psi\left(\frac{\alpha}{4}\right) - \frac{2}{\alpha}\right], \tag{44}$$

where $\Psi(x)$ is the logarithmic derivative of the gamma function.

The results for $\alpha = 0{\cdot}5$, 1, 1·5 and 2 are given in Table I. The comparison between computational and theoretical results shows excellent agreement. In Table II the results for the axially symmetric flow from the truncated cone are tabulated and compared with those obtained by solving the Trefftz integral equation.[5] As compared with Table I, the contraction coefficients of axially symmetric flows are very close to those of plane symmetric flows, as asserted in Gurevich.[5]

4.2. *Flow from a vessel with a funnel-shaped bottom*

We consider the plane symmetric flow from a vessel with a funnel-shaped bottom and with an angle between the bottom and the x-axis of $\pi/4$ (Figure 3), and calculate the contraction coefficients C, varying the ratio d_P of the width $(= 1)$ of the opening to that $(= H)$ of the vessel. For this flow, von Mises calculated C, solving numerically the equations obtained by the hodograph method.[5] The present results agree well with those of von Mises and are seen to approach that $(=0{\cdot}746)$ for the jet from the opening between semi-infinite planes as d_P decreases (Table III).

Table I. Comparison between computational and theoretical values of contraction coefficients for plane symmetric jet from an opening

	α			
	0·5	1	1·5	2
Present results	0·747	0·614	0·541	0·507
Theoretical results	0·746	0·611	0·537	0·500

Table II. Computed contraction coefficients for axially symmetric jet

	α			
	0·5	1	1·5	2
Present results	0·748	0·622	0·549	0·508
Theoretical results	0·75[a]	0·60–0·62[b]	—	0·500[c]

[a, b] Numerical results by Salamatov and Trefftz respectively.[5]
[b] Theoretical result.

Table III. Contraction coefficients for various ratios (d_P, plane symmetric; d_A, axially symmetric) of cross-sectional area of the opening to that of the vessel

		$d_P = d_A^{1/2}$		
		0·1	0·5	0·8
Plane symmetric	Present	0·7906	0·7520	0·7469
	Von Mises	0·789	0·752	0·747
Axially symmetric	Present	0·790	0·754	0·784

For the axially symmetric flow we replace d_P with d_A which is the ratio of the cross-sectional area ($=\pi$) of the opening to that ($=\pi H^2$) of the vessel. As also shown in Table III, the contraction coefficients for axially symmetric flows are nearly identical to those for plane symmetric flows.

5. CONCLUSIONS

We have reduced the minimization problem of describing the jet flow to a variational inequality type of problem in the truncated domain. The variational inequality formulation gives a systematic numerical algorithm based on successive approximations, which can be implemented by using the finite element method. So far as comparison with other results is possible, the results by the present method compare excellently with those by other methods (theoretical or numerical). It is to be noted that in the proposed algorithm the adjustment of the free boundary position can be carried out very simply and there do not appear to be any difficulties in determining the converged solution of the jet problem.

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